A Hybrid Adaptive Fuzzy Control for A Class of Nonlinear MIMO Systems

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Abstract—A hybrid indirect and direct adaptive fuzzy output tracking control schemes are developed for a class of nonlinear multiple-input–multiple-output (MIMO) systems. This hybrid control system consists of observer and other different control components. Using the state observer, it does not require the system states to be available for measurement. Assisted by observer-based state feedback control component, the adaptive fuzzy system plays a dominant role to maintain the closed-loop stability. Being the auxiliary compensation, $H^\infty$ control and sliding mode control are designed to suppress the influence of external disturbance and remove fuzzy approximation error, respectively. Thus, the system performance can be greatly improved. The simulation results demonstrate that the proposed hybrid fuzzy control system can guarantee the system stability and also maintain a good tracking performance.

Index Terms—Adaptive control, fuzzy control, multiple-input–multiple-output (MIMO) nonlinear systems, observer, sliding mode control, stability.

I. INTRODUCTION

FUZZY control methodologies have emerged in recent years as promising ways to approach nonlinear control problems. Fuzzy control, in particular, has had an impact in the control community because of the simple approach it provides to use heuristic control knowledge for nonlinear control problems. In very complicated situations, where the plant parameters are subject to perturbations or when the dynamics of the systems are too complex for a mathematical model to describe, adaptive schemes have to be used online to gather data and adjust the control parameters automatically [1]–[6]. However, no stability conditions have been provided so far for these adaptive approaches. Based on the universal approximation theorem [7], stable direct and indirect adaptive fuzzy control schemes were first developed to control unknown nonlinear systems with the closed stability given by Lyapunov function method [8], [9]. Afterwards, several stable adaptive fuzzy control schemes have been introduced for controlling single-input–single-output (SISO) nonlinear systems [10]–[14]. In these adaptive fuzzy control schemes, the controllers are generally composed of two main components. One is fuzzy logic system for the rough tuning. The other is one kind of robust compensator, such as supervisory control [8], $H^\infty$ control [10], sliding-mode control [11]–[14], or the combination of the latter two, for the fine-tuning. Recently, several stable adaptive fuzzy control schemes are developed for multiple-input–multiple-output (MIMO) nonlinear systems [15]–[20]. However, these adaptive control techniques are only limited to the MIMO nonlinear systems whose states are assumed to be available for measurement. In many practical situations, state variables are often unavailable in nonlinear systems. Thus, the output feedback or observer-based adaptive fuzzy control is required for such complicated applications.

In this paper, indirect and direct adaptive fuzzy controllers are presented for a class of nonlinear MIMO plants with the unavailable state variables based on the previous work [15]–[17]. The adaptive fuzzy logic system is used to approximate the unknown vector functions and then the state observer is constructed, upon which both indirect and direct adaptive fuzzy control system can be developed to control the MIMO system and maintain the system stability. Being the auxiliary compensation, $H^\infty$ control and the sliding mode control are designed to improve the system performance by suppressing the influence of external disturbance and removing the fuzzy approximation error. Thus, the proposed hybrid adaptive fuzzy control system can guarantee the closed-loop stability, and also attenuate the influence of the matching error and external disturbance to an arbitrarily small level.

II. PROBLEM FORMULATION

Assume that a nonlinear MIMO system is represented by the following set of differential equations:

$$
\dot{x}_1 = x_2 \\
\vdots \\
\dot{x}_{(r_1-1)} = x_{r_1} \\
\dot{x}_{r_1} = f_1(x) + \sum_{j=1}^{n} g_{1j}(x)u_j \\
\dot{x}_{(r_1+1)} = x_{(r_1+2)} \\
\vdots \\
\dot{x}_p = f_p(x) + \sum_{j=1}^{n} g_{pj}(x)u_j \\
y_1 = x_1 \\
\vdots \\
y_p = x_{(n-r_p+1)},
$$

(1)
Where \( x = [x_1, ..., x_n]^T \) as the plant state vector; 
\( u = [u_1, ..., u_p]^T \) as the control input; 
\( y = [y_1, ..., y_p]^T \) as the output vector; 
\( f_i, g_{ij}, (i, j) = 1, 2, ..., p \) as the smooth function, 
\( d_j \) is the external disturbance. \( r_1 + r_2 + \cdots + r_p = n \).

Differentiating \( y_1, ..., y_p \) with respect to time for \( r_1, ..., r_p \) times, respectively, until the inputs appear, one obtains the input–output form of (2) as

\[
y^{(r_i)}_i = f_i(x) + \sum_{j=1}^{p} g_{ij}(x)u_j + d_j. \tag{2}
\]

Define

\[
F(x) = [f_1(x), ..., f_n(x)]^T \\
G(x) = [G_1(x), ..., G_p(x)]^T
\]

with

\[
G_i(x) = [g_{i1}(x), ..., g_{ip}(x)]^T \\
\begin{bmatrix}
A &= \text{diag}[A_1, ..., A_p] \\
B &= \text{diag}[B_1, ..., B_p] \\
C^T &= \text{diag}[C_1, ..., C_p] \\
d &= [d_1, ..., d_p]^T.
\end{bmatrix}
\]

Equation (2) can be rewritten as

\[
\dot{x} = Ax + B [F(x) + G(x)u + d] \\
y = CT x. \tag{3}
\]

For the given references \( y_{1m}, ..., y_{pm} \), define the tracking errors as

\[
e_1 = y_1 - y_{1m} \\
e_2 = y_2 - y_{2m} \\
\vdots \\
e_p = y_p - y_{pm}.
\]

Denote

\[
E_1 = [e_1, ..., e_p]^T \\
y_m = [y_{1m}, ..., y_{pm}]^T \\
y_m^{(r)} = [y_{1m}^{(r)}, ..., y_{pm}^{(r)}]^T
\]

and

\[
x = [x_1, ..., x_n]^{(r_1-1)}, ..., x_p, ..., x_{p_n}^{(r_p-1)}]^T \\
y_m = [y_{1m}, ..., y_{1m}^{(r_1-1)}, ..., y_{1m}^{(r_1-1)}]^{(r_p-1)}
\]

then

\[
e = Y_m - x = [e_1, ..., e_1^{(r_1-1)}, ..., e_p, ..., e_p^{(r_p-1)}]^T.
\]

**Assumption 1:** The matrix \( G(x) \) as previously defined is nonsingular, i.e., \( G^{-1}(x) \) exists, and bounded for all \( x \in U_x \), where \( U_x \subset \mathbb{R}^n \) is some compact set.

**Assumption 2:** The plant is feedback linearizable by static-state feedback. It has a general vector \( r = [r_1, ..., r_p]^T \) and zero dynamics are exponentially attractive [22].

**Control Objectives:** Determine a output robust fuzzy controller \( u = u(\hat{x}, E_1(\Theta)) \) and an adaptive law for adjusting the parameter vectors such that the following conditions are met.

i) All the signals involved are uniformly bounded and 
\[
l_i \leq \|e\| \leq L_n \]

ii) For a prescribed attenuation level \( \rho > 0 \), the following \( H^\infty \) tracking performance is achieved as

\[
\int_0^T E^T Q E d\tau \leq \int_0^T E^T Q E d\tau_0 + \frac{1}{\gamma} \Theta^T(0) \Theta(0) + \rho^2 \int_0^T d^T d t. \tag{4}
\]

Where \( Q = Q^T \geq 0, P = P^T \geq 0 \) are weighing matrices, \( \gamma > 0 \) is an adaptation gain, and \( E^T = [\delta^T, e^T] \).

III. OBSERVER-BASED INDIRECT ADAPTIVE FUZZY CONTROL

Let MIMO fuzzy logic systems \( \hat{F}(x|\Theta_1) \) and \( \hat{G}(x|\Theta_2) \) be of the form [17]

\[
\hat{F}(x|\Theta_1) = \Phi(x)\Theta_1 \\
\hat{G}(x|\Theta_2) = \Phi(x)\Theta_2 \tag{5}
\]

where

\[
\Phi(x) = \text{diag}[\xi(x), ..., \xi(x)] \\
\Theta_1 = [\theta_1, ..., \theta_{k1}]^T \\
\Theta_2 = [\theta_{k1+1}, ..., \theta_{k1+k2}]^T \\
\Theta_{1i} = [\theta_{1i}, ..., \theta_{1i}]^T.
\]

If the state variables of (1) are available for measurement, the fuzzy control can be chosen as [16], [17]

\[
u = \hat{G}^{-1}(x|\Theta_2) \left[ -\hat{F}(x|\Theta_1) + y_m^{(r)} + K_c^T e - u_q \right] \tag{6}
\]

where \( \hat{F}(x|\Theta_1) \) and \( \hat{G}(x|\Theta_2) \) are fuzzy systems to approximate \( F(x) \) and \( G(x) \), respectively. \( u_q \) is a \( H^\infty \) robust control to attenuate the disturbance effect on system outputs; and \( K_c^T = [k_{c1}, k_{c2}, ..., k_{c2}] \) is the feedback gain vector to make the characteristic polynomial of \( A - BK_c^T \) to be Hurwitz.

If the state variables are unavailable for measurement, controller (6) can not be used to control nonlinear system (3). Thus, by replacing the system state \( x \) and error \( e \) with their estimates \( \hat{x} \) and \( \hat{e} \), controller will be designed as

\[
u = \hat{G}^{-1}(\hat{x}|\Theta_2) \left[ -\hat{F}(\hat{x}|\Theta_1) + y_m^{(r)} + K_c^T \hat{e} - u_q - u_q \right] \tag{7}
\]

where \( u_q \) is the same as in (6). \( u_q \) is the feedback control for \( \hat{e} \) and need to be designed; and \( u_q \) is a sliding-mode control to compensate fuzzy approximation errors. \( \hat{F}(\hat{x}|\Theta_1) \) and \( \hat{G}(\hat{x}|\Theta_2) \) be of the following form:

\[
\hat{F}(\hat{x}|\Theta_1) = \Phi(\hat{x})\Theta_1 \tag{8}
\]

\[
\hat{G}(\hat{x}|\Theta_2) = \Phi(\hat{x})\Theta_2. \tag{9}
\]
Since $\dot{Y}_m = AY_m - B_0 h_m^{(\theta)} = 0$, substituting (7) into (3) yields
\[
\dot{e} = A\dot{e} - BK_e^T \dot{e} + B(u_a + u_a + u_s) + B[\hat{F}(\hat{x} | \Theta_1) - F(x)] + (\hat{G}(\dot{x} | \Theta_2) - G(x))u - d]
\]
\[
\dot{E}_1 = C^T \dot{e}.
\] (10)

Thus, the problem has been converted to the classical problem of designing a state observer for estimating the state vector $\dot{e}$ in (10).

Design the observer as follows:
\[
\dot{\hat{e}} = A\hat{e} - BK_e^T \hat{e} + K_0(E_1 - \dot{\hat{E}}_1)
\]
\[
\dot{\hat{E}}_1 = C^T \hat{e}.
\] (11)

where $K_0^T = [k_{10}^T, k_{20}^T, \ldots, k_{n_0}^T]$ is the observer gain vector to make sure that the characteristic polynomial of $A - K_0 C^T$ is strictly Hurwitz. Define the observation error as $\tilde{e} = e - \hat{e}$ and subtracting (11) from (10) results in
\[
\dot{\tilde{e}} = (A - K_0 C^T)\tilde{e} + B(u_a + u_a + u_s) + B[(\hat{F}(\hat{x} | \Theta_1) - \hat{F}(\hat{x} | \Theta_1)) + ((\hat{G}(\dot{x} | \Theta_2) - \hat{G}(\dot{x} | \Theta_2))u + w - d]
\]
\[
\dot{\tilde{E}}_1 = C^T \tilde{e}.
\] (16)

According to (12), (13), and (14), (16) can be expressed as
\[
\dot{\tilde{e}} = (A - K_0 C^T)\tilde{e} + B(\theta_1 \hat{e}_2 + \phi(\hat{x}) \Theta_2)u + u_a + u_a + u_s + w - d]
\]
\[
\dot{\tilde{E}}_1 = C^T \tilde{e}.
\] (17)

where $\hat{\Theta}_1 = \Theta_1 - \Theta_1^\phi$, $\hat{\Theta}_2 = \Theta_2 - \Theta_2^\phi$.

**Assumption 3:**

- i) The matrix $\hat{G}(\hat{x} | \Theta_2)$ is invertible, and $d \in L_2$, i.e., $\int_0^\infty d^T \dot{d} dt \leq \infty$
- ii) There exist a constant $\eta > 0$, $D_1 > 0$, $D_2 > 0$, and $M_g > 0$ such that
\[
|\lambda_i(\Delta G(x) + \Delta \hat{G}(\hat{x} | \Theta_2))| \leq \eta
\]
\[
|\Delta F(x) + \Delta \hat{F}(\hat{x} | \Theta_1)| \leq D_1
\]
\[
|\Delta \hat{G}(x) - \Delta \hat{G}(\hat{x} | \Theta_2)| \leq D_2 < 1
\]
\[
\|G^{-1}(\hat{x} | \Theta_2) - \hat{G}(\hat{x} | \Theta_2) + \beta_{10}^\phi + K^T \hat{e}\| \leq M_g.
\]

**Assumption 4:** For the given positive–definite matrices $Q_1$ and $Q_2$, there exist positive–definite solutions $P_1$ and $P_2$ for the matrix equations in (18)–(19), as shown at the bottom of the page, where $\rho$ is a prescribed attenuation level, $R^{-1} = [rI_1, \ldots, rI_p]$ is a matrix, $r$ is a positive parameter to be designed, and $(1 - \eta)R^{-1} - (1/\rho^2)I > 0$.

Since $P_2 B = C$ and $E_1$ is assumed to be measurable, take the control law as
\[
u_a = -\frac{1}{2} \rho R^{-1} B^T P_2 \hat{e}
\]
\[
u_b = -K_{e_1} - P_2 \hat{e}
\]
\[
u_s = -k \beta_{10}^\phi (B^T P_2 \hat{e})
\] (20) (21) (22)

with $k > 0$ as a sliding gain to be determined.

The parameter adaptive adjusting laws are chosen as
\[
\dot{\phi}_1 = -\gamma_1 \Phi(\hat{x}) (B^T P_2 \hat{e})
\]
\[
\dot{\phi}_2 = -\gamma_2 \Phi(\hat{x}) (B^T P_2 \hat{e}_u)
\] (23) (24)

where $\gamma_1 > 0$ and $\gamma_2 > 0$ are two adaptation gains to be designed.

The main property of indirect adaptive fuzzy control scheme is summarized in the following theorem

\[
(A - BK_e^T)^T P_2 + P_2(A - K_0 C^T) - P_2 B(1 - \eta)R^{-1} - \frac{1}{\rho^2} I B^T P_2 + C C^T + Q_2 = 0
\] (18)

\[
\begin{aligned}
(A - BK_e^T)^T P_2 + P_2(A - K_0 C^T) + Q_1 = 0
\end{aligned}
\] (19)
Theorem 1: For the nonlinear system (3), the indirect adaptive fuzzy control law is chosen as (7) with robust and sliding control in (22)–(22) and parameter adaptation in (23) and (24). If Assumptions 3 and 4 are satisfied, then the proposed fuzzy control scheme can guarantee the following properties:

1) \( \dot{x}, x, \dot{e} \in L_{\infty}, \lim_{t \to \infty} E_1 = 0; \)
2) For a prescribed attenuation level \( \rho \), the \( H_{\infty} \) tracking performance (4) is achieved.

The proof of Theorem 1 is given in Appendix A.

Remark 1: Remark 1 is given in Appendix B.

The architecture of the observer-based indirect adaptive fuzzy control system is illustrated in Fig. 1.

IV. OBSERVER-BASED DIRECT ADAPTIVE FUZZY CONTROL

In this section, it is assumed that \( G(x) = G \) is a known constant matrix. If the state variable \( x \) is available for measurement, the direct fuzzy controller is chosen as

\[
u = \hat{u}(x|\Theta) - u_a
\]

where fuzzy logic system \( \hat{u}(x|\Theta) \) aims to directly approximate the following control law:

\[
u^* = G^{-1} \left[ -F(x) + y_m + K_T e \right]
\]

and \( u_a \) is a \( H_{\infty} \) robust control to attenuate effects of both fuzzy approximation errors and the external disturbance.

If the state variable is not available for measurement, the direct fuzzy controller (25) has to be modified as

\[
u = \hat{u}(\hat{x}|\Theta) - u_a - u_b - u_s
\]

where \( u_a \) is a \( H_{\infty} \) robust control, \( u_b \) is a feedback for the estimated errors, and \( u_s \) is a sliding control to cope with the fuzzy approximation errors. Substituting (26) and (27) into (3) results in

\[
\dot{x} = F(x) + G(\hat{u}(\hat{x}|\Theta) - u_a - u_b - u_s)
- Gu^* + Gu^* + d
+ y_m + K_T e - u_a - u_b - u_s
+ G(\hat{u}(\hat{x}|\Theta) - u^*) + d
\]

Equation (28) can be formulated in the form of

\[
\dot{e} = A e - BK_T e + Bu_a + Bu_b + Bu_s + B[\hat{u}(\hat{x}|\Theta) - \hat{u}(\hat{x}|\Theta)] - d
\]

\[
E_1 = C^T e
\]

Design the state observer as

\[
\dot{\hat{e}} = A\hat{e} - BK_T e + K_0(E_1 - \hat{E}_1)
\]

\[
\hat{E}_1 = C^T \hat{e}
\]

where \( K_T = [k_0, k_2, \ldots, k_n] \) is the observer gain vector to guarantee the characteristic polynomial of \( A - K_0C^T \) to be Hurwitz.

If the observation error is defined as \( \hat{e} = e - \hat{e} \), subtracting (30) from (29) results in

\[
\dot{\hat{e}} = (A - K_0C^T)\hat{e} + Bu_a + Bu_b + Bu_s + B[\hat{u}(\hat{x}|\Theta) - \hat{u}(\hat{x}|\Theta)] - d
\]

\[
\hat{E}_1 = C^T \hat{e}
\]

Similarly as in the previous section, fuzzy system \( \hat{u}(\hat{x}|\Theta) \) is expressed as

\[
\hat{u}(\hat{x}|\Theta) = \Phi(\hat{x})\Theta
\]

where \( \hat{x} \) and \( \hat{x} \) are assumed to belong to compact sets \( U_1, U_2 \) and \( \Omega \), respectively

\[
U_1 = \{x \in \mathbb{R}^n : ||x|| \leq M_1\}
U_2 = \{\hat{x} \in \mathbb{R}^n : ||\hat{x}|| \leq M_2\}
\Omega = \{\Theta : ||\Theta|| \leq M\}.
\]

Define the optimal parameter vector \( \Theta^* \)

\[
\Theta^* = \arg\min_{\Theta \in \Omega} \left\{ \sup_{x \in U_1, \hat{x} \in U_2} |u^*(x) - u(\hat{x}|\Theta)| \right\}
\]

and the fuzzy approximation error \( w \),

\[
w = (u^*(x) - \hat{u}(x|\Theta^*)) + (\hat{u}(x|\Theta^*) - (\hat{u}(\hat{x}|\Theta^*))
\]

The dynamic equation of the observation error (31) can be written as

\[
\dot{\hat{e}} = (A - K_0C^T)\hat{e} + Bu_a + Bu_b + Bu_s + B[\hat{u}(\hat{x}|\Theta^*) - \hat{u}(\hat{x}|\Theta)] + w - d
\]

\[
\hat{E}_1 = C^T \hat{e}
\]

According to (31), (34) can be expressed as

\[
\dot{\hat{e}} = (A - K_0C^T)\hat{e} + B[\hat{u}(\hat{x}|\Theta^*) + \hat{u}(\hat{x}|\Theta)] + u_a + u_b + u_c + w - d
\]

\[
\hat{E}_1 = C^T \hat{e}
\]

where \( \hat{\Theta} = \Theta^* - \Theta \) is an error parameter.
Assumption 5: There exist a constant $k$ such that $|u| \leq k$, and $d \in L_2$, i.e., $\int_0^\infty \|d\|^2 dt < \infty$.

Assumption 6: For the given positive–definite matrices $Q_1$ and $Q_2$, there exist positive–definite matrixes $P_1$ and $P_2$ for the (37) and (38), respectively; see (37)–(38), as shown at the bottom of the page. The direct adaptive control scheme is designed as

\begin{align*}
    u &= \hat{u}(\hat{x}|\Theta) - u_a - u_b - u_s, \\
    u_a &= -\frac{1}{2} R^{-1} B^T P_2 \hat{e} \\
    u_b &= -K_0^T P_1 \hat{e} \\
    u_s &= -k \text{sgn}(B^T P_2 \hat{e}) \\
    \hat{\Theta} &= \gamma \Phi(\hat{x})^T C (B^T P_2 \hat{e})
\end{align*}

where $\rho$ is a presribed attenuation level, $R^{-1} = [r I_3, \ldots, r I_p]$ is a matrix, $r$ is a positive parameter to be designed, and $R^{-1} - (1/\rho^2) I > 0$.

Theorem 2: For the nonlinear MIMO system (3), the fuzzy controller law is chosen as (39)–(42) with parameter adjusting law (43). If Assumptions 5 and 6 are satisfied, the direct adaptive fuzzy control scheme proposed can guarantee the following properties:

1) $\dot{\hat{x}}, x, \dot{e} \in L_{\infty}$, $\lim_{t \rightarrow \infty} E_1 = 0$;
2) the $H_\infty$ tracking performance (4) is achieved for a prescribed attenuation level $\rho$.

The proof of Theorem 2 is given in Appendix C.

Remark 2: Remark 2 is similar to Remark 1 that is given in Appendix B.

The architecture of the observer-based direct adaptive fuzzy control system is illustrated in Fig. 2.

V. SIMULATION

A. Observer-Based Indirect Adaptive Fuzzy Control Approach

Consider the nonlinear differential equation given by

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix}
= \begin{bmatrix}
x_2 \\
x_1 + x_2 + x_3 \\
x_1 + 2x_2 + 3x_3 x_1
\end{bmatrix}
+ \begin{bmatrix}
0 \\
3u_1 + u_2 \\
u_1 + 2(2 + 0.5\sin(x_1))u_2
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0.5e^{-t}\sin(t) \\
0.5e^{-t}\sin(t)
\end{bmatrix}
$$

$$
y_1 = x_1, \\
y_2 = x_2.
$$

Rewrite (44) in the following form:

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0\end{bmatrix} \begin{bmatrix} x_1 \\
x_2 \\
x_3\end{bmatrix} + \begin{bmatrix} 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0\end{bmatrix} \begin{bmatrix} x_1 + x_2 + x_3 \\
x_1 + 2x_2 + 3x_3 x_1 + 3 + 1 + 4 + \sin(x_1) \\
0.5e^{-t}\sin(t) \\
0.5e^{-t}\sin(t)\end{bmatrix}
$$

where relative degree $r = [r_1 r_2] = [2 1]$

$$
A = \text{diag}(A_1, A_2), B = \text{diag}(B_1, B_2), \\
C = \text{diag}(C_1, C_2), A_1 = \begin{bmatrix} 0 & 1 \\
0 & 0\end{bmatrix}, B_1 = \begin{bmatrix} 0 \\
1\end{bmatrix}, C_1 = [1, 0], \\
A_2 = 0, B_2 = 1, C_2 = 1; \\
F(x) = \begin{bmatrix} x_1 + x_2 + x_3 \\
x_1 + 2x_2 + 3x_3 x_1 + 3 + 1 + 4 + \sin(x_1) \\
0.5e^{-t}\sin(t) \\
0.5e^{-t}\sin(t)\end{bmatrix}, \\
G(x) = \begin{bmatrix} 0 \\
3 \\
1 \\
4 + \sin(x_1)\end{bmatrix}, \quad d = \begin{bmatrix} 0 \\
0.5e^{-t}\sin(t) \\
0.5e^{-t}\sin(t)\end{bmatrix}.
$$

The feedback and observer gain matrices are chosen as

$$
K_c = \begin{bmatrix} 1 & 1 & 0 \\
0 & 0 & 1\end{bmatrix}, K_0^T = \begin{bmatrix} 60 & 900 & 0 \\
0 & 0 & 10\end{bmatrix}.
$$

$$
\begin{align*}
(A - BK_c^T)^T P_1 + P_2 A - BK_c^T + P_1 & = 0 \\
(A - K_0 C)^T P_2 + P_2 (A - K_0 C) - P_2 B [R^{-1} - \frac{1}{\rho^2} I] B^T P_2 + CCC^T + Q_2 & = 0
\end{align*}
$$

(37)
Fig. 3. Trajectories of state variables and their estimates for indirect adaptive fuzzy control (a) State $x_1$ and its estimation $\hat{x}_1$. (b) State $x_2$ and its estimation $\hat{x}_2$. (c) State $x_3$ and its estimation $\hat{x}_3$.

Membership functions are chosen as the following:

\[
\begin{align*}
\mu_{F_1}(x_i) &= \frac{1}{1 + \exp(5 \times (x_i + 0.8))} \\
\mu_{F_2}(x_i) &= \exp[-(x_i + 0.6)^2] \\
\mu_{F_3}(x_i) &= \exp[-(x_i + 0.4)^2] \\
\mu_{F_4}(x_i) &= \exp[-x_i^2] \\
\mu_{F_5}(x_i) &= \exp[-(x_i - 0.4)^2] \\
\mu_{F_6}(x_i) &= \exp[-(x_i - 0.6)^2] \\
\mu_{F_7}(x_i) &= \frac{1}{1 + \exp(-5 \times (x_i - 0.8))} \\
\end{align*}
\]

\[i = 1, 2, 3.\]

Defining seven fuzzy rules to be of the following form:

\[R^{(j)}: \text{If } x_1 \text{ is } F_1^j \text{ and } x_2 \text{ is } F_2^j \text{ and } x_3, \text{ then } y \text{ is } G_j^{(j)} \quad (j = 1, 2, \ldots, 7)\]
Denote

\[
D = \sum_{j=1}^{7} \prod_{i=1}^{5} \mu_{Fi}(x_i)
\]

\[
\dot{x}(x) = \left( \prod_{i=1}^{5} \mu_{F_i}(x) / D \right)^T \theta = [\theta_1, \ldots, \theta_7]^T
\]

In the same way as [17], onstructing fuzzy systems \( \hat{F}(x|\Theta_1) \) and \( \hat{G}(x|\Theta_2) \) and by using observer (17), we obtain the estimated \( \hat{x} \), thus, we have fuzzy systems \( \hat{F}(x|\Theta_1) \) and \( \hat{G}(x|\Theta_2) \), respectively.

Let \( \eta = 0 \), \( D_1 = 0.5 \), \( D_2 = 0.1 \), \( M_\eta = 3 \), \( k = 6 \). Adaptation adjusting factors are chosen as \( \gamma_1 = 0.2 \), and \( \gamma_2 = 10 \). The initial conditions are chosen as

\[
x_1(0) = x_2(0) = x_3(0) = 0.1; \\
x_1(0) = \hat{x}_2(0) = \hat{x}_3(0) = 0.01 \\
\Theta_1(0) = 0, \Theta_2(0) = 0.1 I_{21 \times 1}
\]

and the tracking reference signals as

\[
y_{1m} = \frac{\pi}{30} \sin t, \ y_{2m} = \frac{\pi}{30} \cos t.
\]

The simulation results are shown in Figs. 3 and 4. A good tracking performance can be achieved as shown in Fig. 3 where a perfect estimate of states is obtained.

**B. Observer-Based Direct Adaptive Fuzzy Approach**

Consider the following nonlinear systems:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
+ \begin{bmatrix}
0 & 0.5e^{-t} \sin(t) \\
0 & 0.5e^{-t} \sin(t)
\end{bmatrix}.
\]

The fuzzy basis functions and fuzzy logic system are determined similar to those in Section V-A. The tracking reference signals are chosen as the same as those in Section V-A. The initial conditions are chosen as

\[
x_1(0) = x_2(0) = x_3(0) = 0.1 \\
\hat{x}_1(0) = \hat{x}_2(0) = \hat{x}_3(0) = 0.01 \\
\Theta(0) = 0, \gamma = 0.1, k = 3.
\]

Select

\[
K_c = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad K_0 = \begin{bmatrix}
60 & 900 & 0 \\
0 & 0 & 10
\end{bmatrix}
\]

\[
L_1^{-1}(s) = \frac{1}{s + 2} \quad \text{and} \quad L_2^{-1}(s) = 1.
\]

For the given prescribed attenuation level \( \rho = \sqrt{\bar{\rho}} \), \( \bar{\rho} \) and definite matrices \( Q_1 = Q_2 = \diag[10, 10] \), positive–definite matrices are solved from (37) and (38)

\[
P_1 = \begin{bmatrix}
12.5 & 2.5 & 0 \\
2.5 & 3.75 & 0 \\
0 & 0 & 5
\end{bmatrix}, \quad P_2 = \begin{bmatrix}
75 & -5 & 0 \\
-5 & 0.416 & 0 \\
0 & 0 & 0.5
\end{bmatrix}
\]

\[
P_1 = \begin{bmatrix}
12.5 & 2.5 & 0 \\
2.5 & 3.75 & 0 \\
0 & 0 & 5
\end{bmatrix}, \quad P_2 = \begin{bmatrix}
75 & -5 & 0 \\
-5 & 0.416 & 0 \\
0 & 0 & 0.5
\end{bmatrix}
\]

The simulation results are illustrated in Figs. 5 and 6. A good tracking performance can be achieved as shown in Fig. 5 where a perfect estimate of states is obtained.
VI. CONCLUSION

Both indirect and direct adaptive fuzzy controls in hybrid structure are proposed for a class of nonlinear MIMO processes. Using the state observer, it does not require the system states to be available for measurement. Aided by observer-based state feedback control, the adaptive fuzzy system behaves a primary control system to maintain a stable control of the
MIMO process. Being the auxiliary compensation, the $H^\infty$ control and the sliding mode control components are designed to improve the system performance without disturbing the original stability. The $H^\infty$ control will suppress the influence from external disturbance and the sliding control will remove the fuzzy approximation error. The proposed hybrid adaptive fuzzy system not only guarantees the closed-loop stability, but also achieves the desired $H^\infty$ tracking performance.

APPENDIX A

A. Proof of Theorem 1

Consider the Lyapunov function candidate

$$V = \frac{1}{2} \dot{e}^T P_1 \dot{e} + \frac{1}{2} \dot{e}^T P_2 \dot{e} + \frac{1}{2 \gamma_1} \dot{\hat{O}}^T \dot{\hat{O}}_1 + \frac{1}{2 \gamma_2} \text{tr}(\dot{\hat{O}}_2^T \dot{\hat{O}}_2).$$  (A1)

The time derivative of $V$ is

$$\dot{V} = \frac{1}{2} \dot{e}^T P_1 \dot{e} + \frac{1}{2} \dot{e}^T P_2 \dot{e} + \frac{1}{2 \gamma_1} \dot{\hat{O}}^T \dot{\hat{O}}_1 + \frac{1}{\gamma_2} \text{tr}(\dot{\hat{O}}_2^T \dot{\hat{O}}_2).$$  (A2)

Substituting (11) and (19) into (A2) yields

$$\dot{V} \leq \frac{1}{2} \dot{e}^T [(A - BK_C^T)^T P_1 + P_1 (A - BK_C^T)] \dot{e} + e^T P_1 K_0 C \dot{e} + e^T P_2 B u_b + \frac{1}{2} \dot{e}^T P_2 \dot{e} + \frac{1}{2 \gamma_1} \dot{\hat{O}}_1^T \dot{\hat{O}}_1 + \frac{1}{\gamma_2} \text{tr}(\dot{\hat{O}}_2^T \dot{\hat{O}}_2).$$  (A3)

Via parameter adaptive law (23) and (24), (A3) can be formulated as

$$\dot{V} \leq \frac{1}{2} \dot{e}^T [(A - BK_C^T)^T P_1 + P_1 (A - BK_C^T)] \dot{e} + e^T P_1 K_0 C \dot{e} + e^T P_2 B u_b + \frac{1}{2} \dot{e}^T P_2 \dot{e} + \frac{1}{2 \gamma_1} \dot{\hat{O}}_1^T \dot{\hat{O}}_1 + \frac{1}{\gamma_2} \text{tr}(\dot{\hat{O}}_2^T \dot{\hat{O}}_2).$$  (A4)

Since

$$\frac{1}{2} \dot{e}^T [(A - BK_C^T)^T P_1 + P_1 (A - BK_C^T)] \dot{e} + e^T P_1 K_0 C \dot{e} + e^T P_2 B u_b + \frac{1}{2} \dot{e}^T P_2 \dot{e} + \frac{1}{2 \gamma_1} \dot{\hat{O}}_1^T \dot{\hat{O}}_1 + \frac{1}{\gamma_2} \text{tr}(\dot{\hat{O}}_2^T \dot{\hat{O}}_2) \leq D_1 + D_2 M_g + k D_2. $$  (A5)

and

$$\dot{V} \leq \frac{1}{2} \dot{e}^T P_2 B [(\Delta G(\dot{x}) + \Delta G(x)) \dot{G}^{-1}(\dot{x}) \Theta_2] u_a$$

$$= -\frac{1}{2} \dot{e}^T P_2 B [(\Delta G(\dot{x}) + \Delta G(x)) \dot{G}^{-1}(\dot{x}) \Theta_2] R^{-1} B^T P_2 \dot{e}$$

$$\leq \frac{1}{2} \dot{e}^T P_2 B \eta R^{-1} B^T P_2 \dot{e}.$$  (A6)

Substituting (A5) and (A6) into (A4) results in

$$\dot{V} \leq \frac{1}{2} \dot{e}^T [(A - BK_C^T)^T P_1 + P_1 (A - BK_C^T)] \dot{e} + e^T P_1 K_0 C \dot{e} + e^T P_2 B u_b + \frac{1}{2} \dot{e}^T P_2 \dot{e} + \frac{1}{2 \gamma_1} \dot{\hat{O}}_1^T \dot{\hat{O}}_1 + \frac{1}{\gamma_2} \text{tr}(\dot{\hat{O}}_2^T \dot{\hat{O}}_2) + \Delta F(\dot{x}) + \Delta F(x) \dot{G}^{-1}(\dot{x}) \Theta_2] u_a$$

$$\leq \frac{1}{2} \dot{e}^T [(A - BK_C^T)^T P_1 + P_1 (A - BK_C^T)] \dot{e} + e^T P_1 K_0 C \dot{e} + e^T P_2 B u_b + \frac{1}{2} \dot{e}^T P_2 \dot{e} + \frac{1}{2 \gamma_1} \dot{\hat{O}}_1^T \dot{\hat{O}}_1 + \frac{1}{\gamma_2} \text{tr}(\dot{\hat{O}}_2^T \dot{\hat{O}}_2) + \Delta F(\dot{x}) + \Delta F(x) \dot{G}^{-1}(\dot{x}) \Theta_2] u_a.$$  (A7)
Choosing $k = (D_1 + D_2M_0/1 - D_2)$ and using Lyapunov (18) and Riccati-like (19), (A7) becomes

\[
\dot{V} \leq -\frac{1}{2} \varepsilon Q_1 \varepsilon - \frac{1}{2} \varepsilon^T Q_2 \varepsilon - \frac{1}{2} \rho \varepsilon^T P_2 B B^T P_2 \varepsilon
\]

\[
+ \frac{1}{2} (d^T B^T P_2 \dot{e} + e^T P_2 B d)
\]

\[
= -\frac{1}{2} \varepsilon^T Q_1 \varepsilon - \frac{1}{2} \varepsilon^T Q_2 \varepsilon - \frac{1}{2} \left( \frac{1}{\rho} B^T P_2 \varepsilon - \rho d \right)^T
\]

\[
\cdot \left( \frac{1}{\rho} B^T P_2 \varepsilon - \rho d \right) + \frac{1}{2} \rho^2 d^T d.
\]  

(A8)

Denoting $Q = \text{diag}[Q_1, Q_2]$ and $E_T = [\varepsilon^T, \dot{e}]$, then the inequality (A8) becomes

\[
\dot{V} \leq -\frac{1}{2} E_T^T Q E + \frac{1}{2} \rho^2 d^T d.
\]  

(A9)

Since $d \in L_2$, it is included that $\varepsilon, \dot{e}, \dot{x}, x, u \in L_\infty$ and

\[
\lim_{t \to \infty} E_T = 0 \quad \text{[8]}. \quad \text{Thus, there are} \quad \lim_{t \to \infty} \dot{e} = 0 \quad \text{and} \quad \lim_{t \to \infty} \dot{E}_1 = 0.
\]

Integrating inequality (A9) from $t = 0$ to $t = T$ yields

\[
V(T) - V(0) \leq -\frac{1}{2} \int_0^T E_T^T Q E dt + \frac{1}{2} \rho^2 \int_0^T d^T dt.
\]  

(A10)

Since $V(T) \geq 0$, inequality (A10) implies

\[
\frac{1}{2} \int_0^T E_T^T Q E dt \leq \frac{1}{2} H^T(0) P H(0) + \frac{1}{2} \gamma \Theta_1^T(0) \Theta_1(0)
\]

\[
+ \frac{1}{2} \gamma \Theta_2^T(0) \Theta_2(0) + \frac{1}{2} \rho^2 \int_0^T d^T dt.
\]  

(A11)

The $H^\infty$ tracking performance (4) is achieved for a prescribed attenuation level $\rho > p$.

**APPENDIX B**

**Remark 1:** Since $A - BK^T_C$ is a Hurwitz matrix with a properly chosen $K_\infty$, there exists a positive–definite matrix $P_1$ for (18) in Assumption 4. Actually, Theorem 1 depends on whether there exists a positive–definite matrix $P_2$ for Riccati-like (19).

- If a positive–definite $P_2$ exists for (19), we have $B^T P_2 \varepsilon = \tilde{E}_1$. The states estimate $\hat{e}$ and $\tilde{E}_1$ are available to make the proposed adaptive fuzzy control scheme realizable.
- If a positive–definite $P_2$ does not exist for (19), then (17) $(A, B, C)$ can be converted to the strictly positive-real (SPR) system by using the same way as in [21]. The details are as follows.

First, the output error dynamics of (17) can be formulated as

\[
\dot{E}_1 = H(s)[\Phi(\hat{x})\Theta_1 + \Phi(\hat{x})\Theta_2 u + u + u_a + u_b + u_s + w - d].
\]  

(B1)

where

\[
H(s) = C^T(sI - (A - K_0 C^T))^{-1} B.
\]  

(B2)

The transfer function $H(s)$ is a known stable transfer function matrix. In order to use the SPR-Lyapunov design approach, (B1) can be written as

\[
\dot{E}_1 = H(s)L(s)[L^{-1}(s)\Phi(\hat{x})\Theta_1 + L^{-1}(s)\Phi(\hat{x})\Theta_2 u
\]

\[
+ L^{-1}(s) (u_a + u_b + u_s + w - d)],
\]  

(B3)

with

\[
L(s) = \text{diag}[L_1(s), \ldots, L_p(s)]
\]  

and

\[
L_i(s) = b_{i1}s^{m_i-1} + \cdots + b_{i1} (m_i < r_i), i = 1, 2, \ldots, p.
\]

$L(s)$ is chosen to be a proper stable transfer function matrix and to make $H(s)L(s)$ to be a proper SPR transfer function matrix. Then the state-space realization of (B4) can be written as

\[
\dot{\hat{e}}_c = (A - K_0 C^T)\hat{e}_c + B_c [\Phi_1(\hat{x})\Theta_1 + \Phi_1(\hat{x})\Theta_2 u + u_a + u_b + u_s + w - d]
\]  

(B4)

where

\[
B_c = \text{diag}[B_{c1}, \ldots, B_{cp}]
\]

\[
B_{ci} = [0, 0, \ldots, b_{i1}, \ldots, b_{ni}] \quad i = 1, 2, \ldots, p.
\]

After this transformation, $(A - K_0 C^T, B_C, C)$ is SPR systems. Let $(1 - \eta) R^{-1} = ((1/\rho^2) + 1) I$ to change Riccati-like (19) to be

\[
(A - K_0 C^T)^T P_2 + P_2 (A - K_0 C^T) + Q_2 = 0
\]

\[
P_2 B_c = C.
\]  

(B6)

with

\[
Q_2 = C^T C + Q_2.
\]

Since $(A - K_0 C^T, B, C)$ is SPR, for the given positive matrix $Q_2$, a positive–definite solution $P_2$ exists for (B6).

**APPENDIX C**

**A. Proof of Theorem 2**

Consider Lyapunov function as

\[
V = \frac{1}{2} e^T P_1 e + \frac{1}{2} \dot{e}^T P_2 \dot{e} + \frac{1}{2} \gamma \dot{\Theta}^T \dot{\Theta}.
\]  

(C1)

The derivative of $V$ is

\[
\dot{V} = \frac{1}{2} e^T (A - BK^T_C) P_1 + P_1 (A - BK^T_C) \dot{e}
\]

\[
+ \frac{1}{2} \dot{e}^T P_1 K_0 C^T \dot{e} + \frac{1}{2} \dot{e}^T P_2 B \dot{u}
\]

\[
+ \frac{1}{2} \dot{e}^T [A - K_0 C^T]^T P_2 + P_2 (A - K_0 C^T) \dot{e}
\]

\[
+ \frac{1}{2} \dot{e}^T P_1 K_0 C^T \dot{e} + \frac{1}{2} \dot{e}^T P_2 B \dot{u}
\]

\[
+ \frac{1}{2} \dot{e}^T P_1 K_0 C^T \dot{e} + \frac{1}{2} \dot{e}^T P_2 B \dot{u} + \frac{1}{2} \dot{e}^T P_2 B \dot{u} + \frac{1}{2} \dot{e}^T P_2 B \dot{u} + \frac{1}{2} \dot{e}^T P_2 B \dot{u}.
\]  

(C2)

Substituting (30) and (35) into (C2) yields

\[
\dot{V} = \frac{1}{2} e^T [(A - BK^T_C)^T P_1 + P_1 (A - BK^T_C)] \dot{e}
\]

\[
+ \frac{1}{2} \dot{e}^T P_1 K_0 C^T \dot{e} + \frac{1}{2} \dot{e}^T P_2 B \dot{u}
\]

\[
+ \frac{1}{2} \dot{e}^T [A - K_0 C^T]^T P_2 + P_2 (A - K_0 C^T) \dot{e}
\]

\[
+ \frac{1}{2} \dot{e}^T P_1 K_0 C^T \dot{e} + \frac{1}{2} \dot{e}^T P_2 B \dot{u} + \frac{1}{2} \dot{e}^T P_2 B \dot{u} + \frac{1}{2} \dot{e}^T P_2 B \dot{u}.
\]  

(C3)
Substituting sliding control (42) and parameter adaptive law (43) into (C3) results in
\[
\dot{V} \leq \frac{1}{2} \dot{\varepsilon}^T [(A - BK)^T P_1 + P_1 (A - BK^T)] \dot{\varepsilon}
+ \frac{1}{2} \dot{\varepsilon}^T [(A - K_0 C)^T P_2 + P_2 (A - K_0 C^T)] \dot{\varepsilon}
- P_2 B R^{-1} B^T P_2 + CC^T \dot{\varepsilon}
+ \dot{\varepsilon}^T P_2 B w - k C^T P_2 B \text{sgn}(\dot{\varepsilon}^T P_2 B) - \dot{\varepsilon}^T P_2 B d
\leq \frac{1}{2} \dot{\varepsilon}^T [(A - BK)^T P_1 + P_1 (A - BK^T)] \dot{\varepsilon}
+ \frac{1}{2} \dot{\varepsilon}^T [(A - K_0 C)^T P_2 + P_2 (A - K_0 C^T)] \dot{\varepsilon}
- P_2 B R^{-1} B^T P_2 + CC^T \dot{\varepsilon}
+ \frac{1}{2} (d^T P_2 \dot{\varepsilon} + \dot{\varepsilon}^T P_2 d)
+ \left(\|u\| - k\right) \sum_{i=1}^{p} |(\dot{\varepsilon}^T P_2 B)_i|.
\]
(C4)

From (42), (37), (38), and Assumption 5, inequality (C5) becomes
\[
\dot{V} \leq - \frac{1}{2} Q_1 \dot{\varepsilon} - \frac{1}{2} \dot{\varepsilon}^T Q_2 \dot{\varepsilon}
- \frac{1}{2} \rho^2 \dot{\varepsilon}^T P_2 B T P_2 \dot{\varepsilon}
+ \frac{1}{2} \rho \dot{\varepsilon}^T P_2 C d
- \frac{1}{2} \rho \dot{\varepsilon}^T P_2 \dot{\varepsilon}
= - \frac{1}{2} \dot{\varepsilon}^T Q_1 \dot{\varepsilon} - \frac{1}{2} \dot{\varepsilon}^T Q_2 \dot{\varepsilon}
- \frac{1}{2} \left( \frac{1}{\rho} B^T P_2 \dot{\varepsilon} - \rho d \right)^T \cdot \left( \frac{1}{\rho} B^T P_2 \dot{\varepsilon} - \rho d \right)
+ \frac{1}{2} \rho^2 \dot{\varepsilon}^T d.
\]
(C5)

Denoting \( Q = \text{diag}[Q_1, Q_2] \), and \( E^T = [\dot{\varepsilon}^T, \dot{\varepsilon}] \), (C5) becomes
\[
\dot{V} \leq - \frac{1}{2} E^T Q E + \frac{1}{2} \rho^2 d^T d.
\]
(C6)

Similar to the proof of Theorem 1, we can complete the proof of Theorem 2.

REFERENCES